A Simple Algorithm for Worst Case Optimal Join and Sampling

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Joining relations

Joining like a pro

$$Q := R(x_1, x_2) \wedge S(x_1, x_3) \wedge T(x_2, x_3)$$

- Devise a query plan: $(R \bowtie S) \bowtie T$
- Materialize the intermediate joins.

R x_1	x_2
0	0
0	1
2	1

S x_1	x_3
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0	2
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T x_2	x_3
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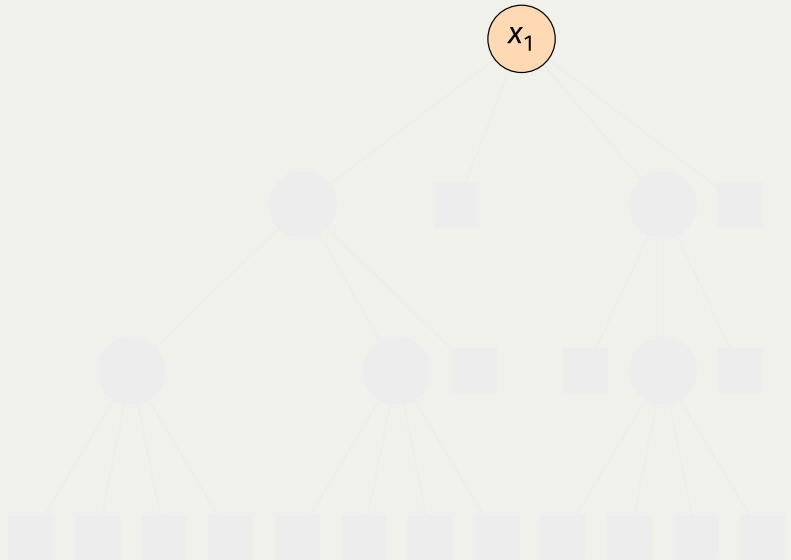

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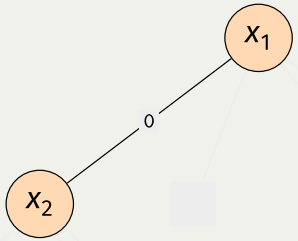

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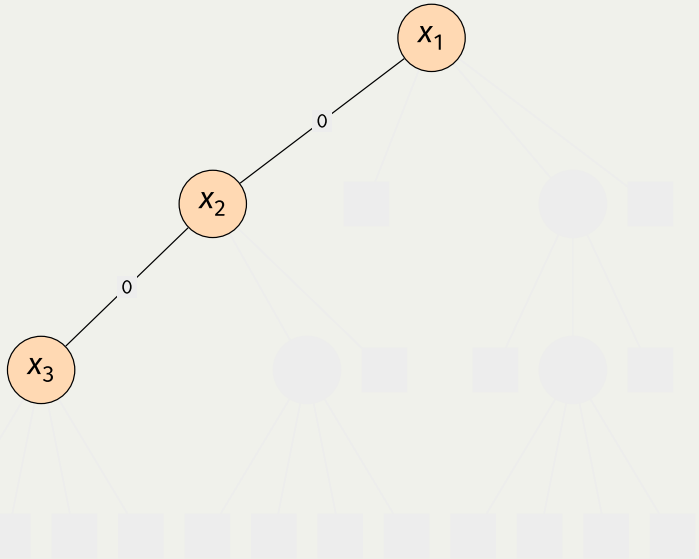

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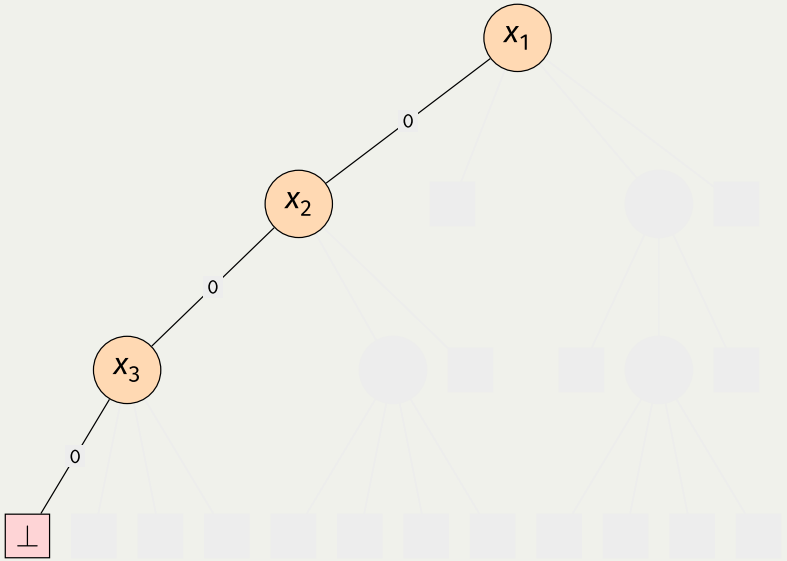

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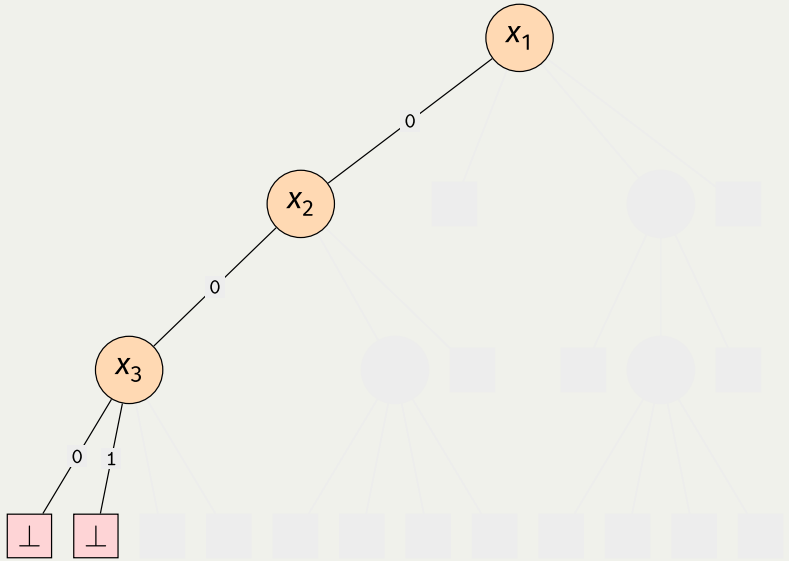

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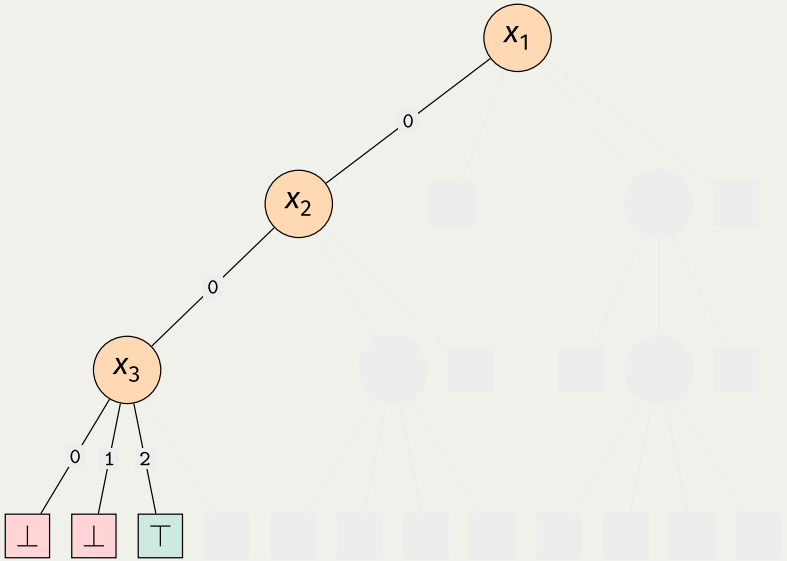

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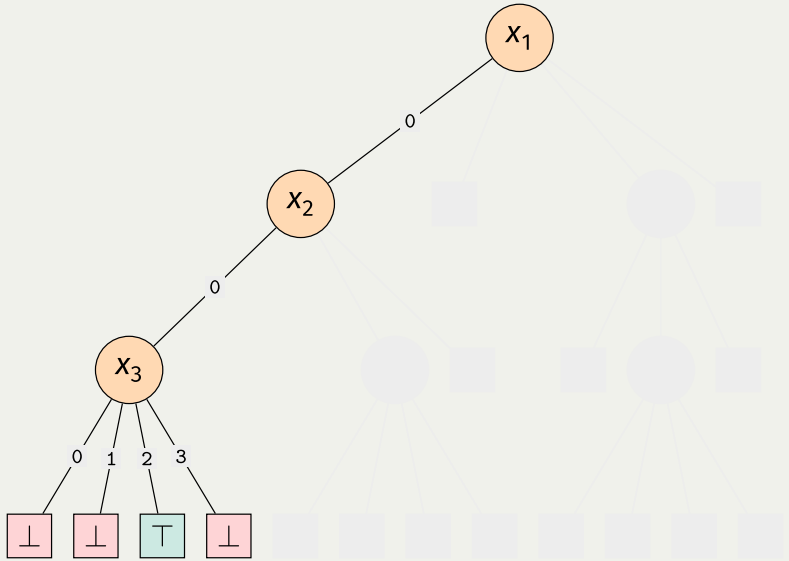

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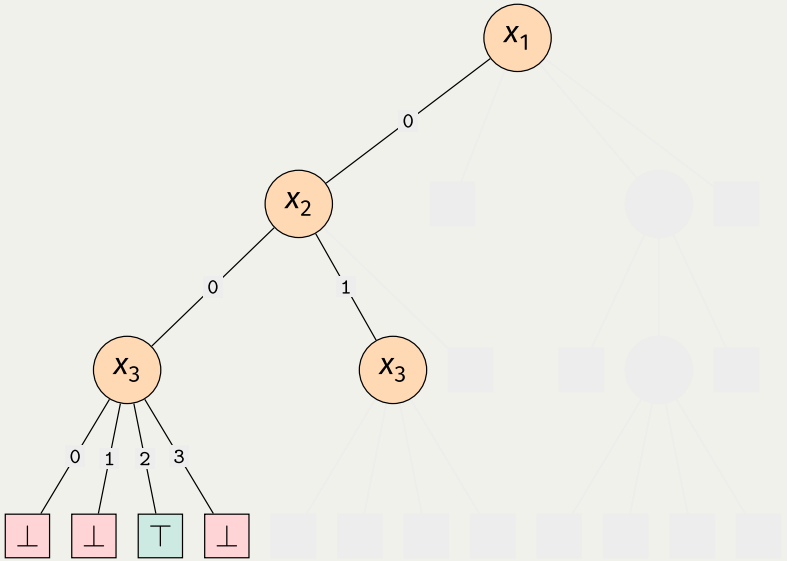

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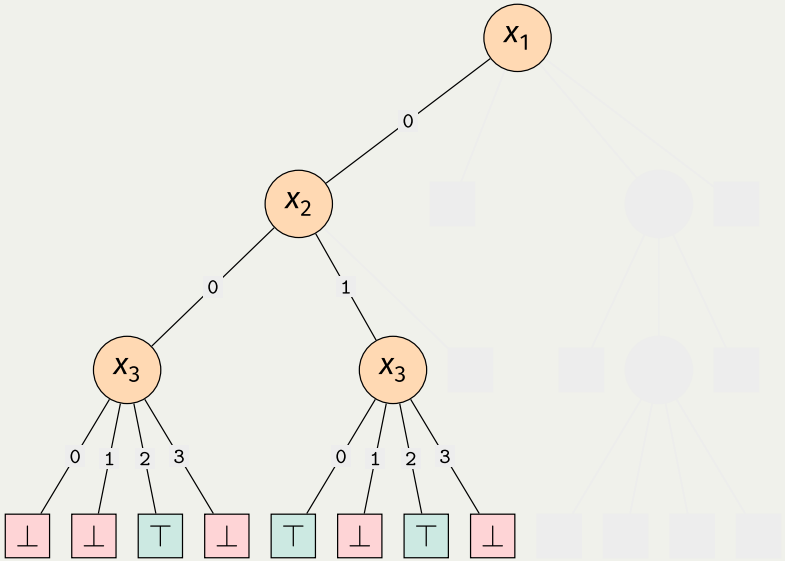

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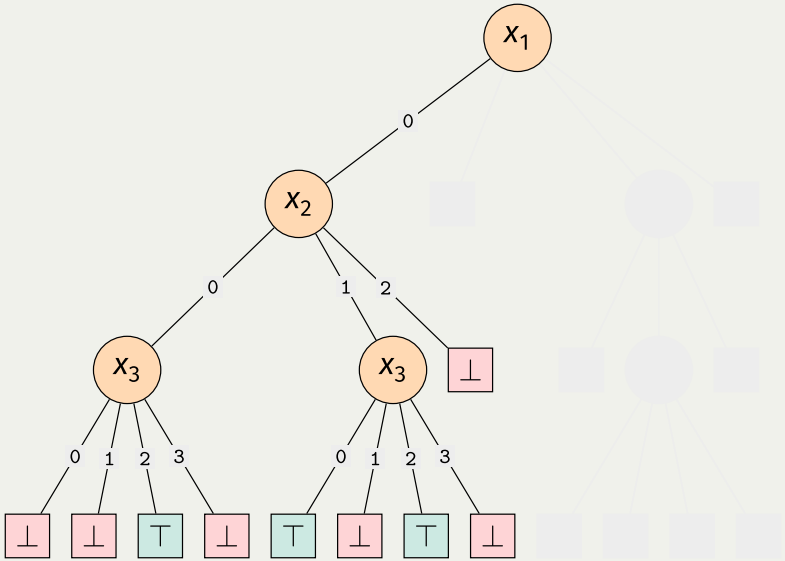

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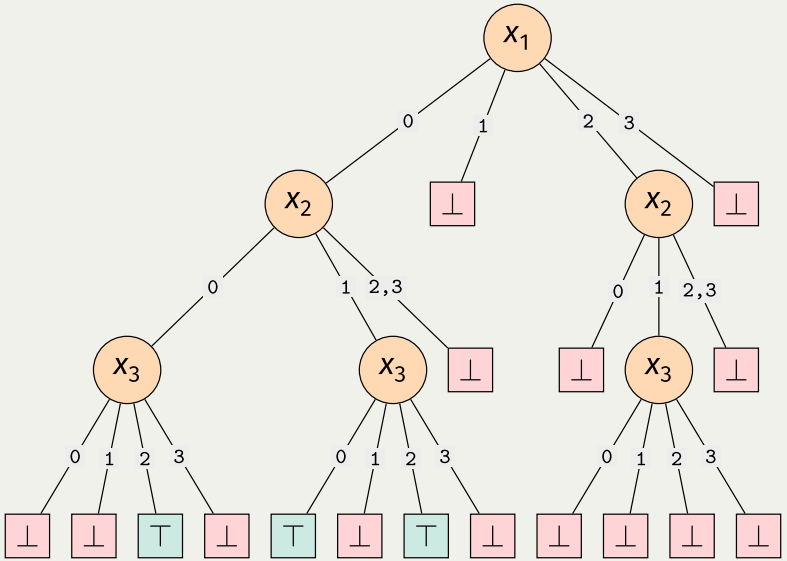

Disruptive poll

In theory, is it better to join like:

a. A pro
b. A brute

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No confidence vote

a. Go home, you are not qualified to talk about joins after saying dumb things like that.
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What is wrong with joining like a pro

$$Q := R(x_1, x_2) \wedge S(x_1, x_3) \wedge T(x_2, x_3)$$

- It is known that if $|R^{\mathbb{D}}|, |S^{\mathbb{D}}|, |T^{\mathbb{D}}| \leq N$, then $|Q(\mathbb{D})| \leq N^{1.5}$.
- $R \bowtie S$ may have N^2 answers!

Worst scenario for query plans

Consider $\mathbb{D}$ on domain $D = D_1 \uplus D_2 \uplus D_3$ with:

- $0 \notin D$
- $|D_1| = |D_2| = |D_3| = N$.

R	x_1	x_2
	0	D_2
	D_1	0

S	x_1	x_3
	0	D_3
	D_1	0

T	x_2	x_3
	0	D_3
	D_2	0

- $|R^{\mathbb{D}} \bowtie S^{\mathbb{D}}| \geq N^2$, $|R^{\mathbb{D}} \bowtie T^{\mathbb{D}}| \geq N^2$, $|S^{\mathbb{D}} \bowtie T^{\mathbb{D}}| \geq N^2$
- $|Q(\mathbb{D})| = 0$.

Every query plan will materialize a table of size $O(N^2)$ but the answer table will never be of size greater than $(2N)^{1.5}$.

And the brute?

Domain $D = D_1 \uplus D_2 \uplus D_3$ with $0 \notin D$ and $|D_1| = |D_2| = |D_3| = N$.

R	x_1	x_2
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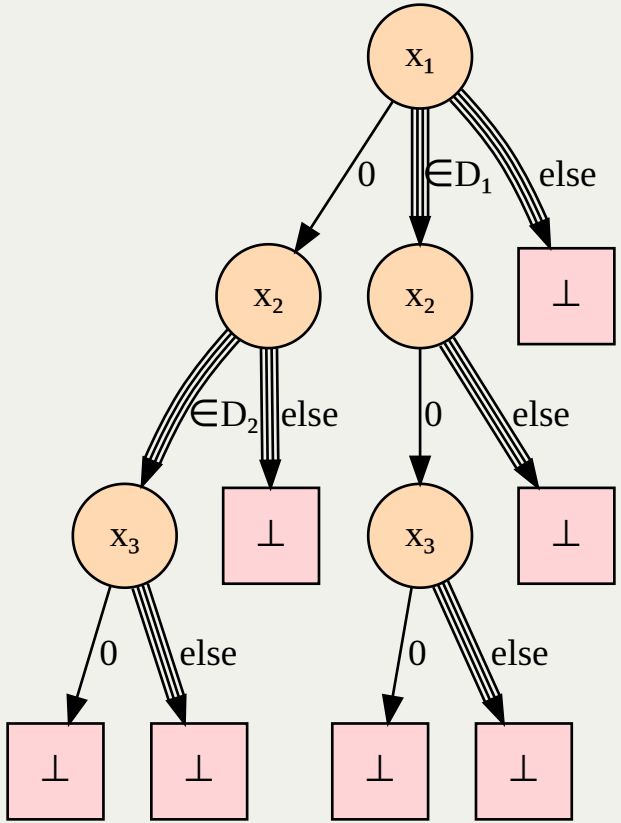

If we do the "else" branches efficiently (e.g. by reading values from one table), the algorithm makes $O(N)$ recursive calls.

Worst-case optimality

Worst-case optimal join

$$Q := R(x_1, x_2) \wedge S(x_1, x_3) \wedge T(x_2, x_3)$$

Ideal complexity: output $Q(\mathbb{D})$ in time $O(f(|Q|) \cdot |Q(\mathbb{D})|)$…

… unlikely to be possible.

$f(|Q|)$: data complexity, ie, Q is considered constant. Ideally, f is a reasonable polynomial though.

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… unlikely to be possible.

Worst case optimal: output $Q(\mathbb{D})$ in time $\tilde{O}(f(|Q|) \cdot N^{1.5})$.

N is the size of the largest input relation and $\tilde{O}(\cdot)$ ignores polylog factors.

$f(|Q|)$: data complexity, ie, Q is considered constant. Ideally, f is a reasonable polynomial though.

Worst case value

Consider a join query Q and all databases for Q with a bound N on the table size:

$$\mathcal{D}_Q^{\leq N} = \{\mathbb{D} \mid \forall R \in Q, |R^{\mathbb{D}}| \leq N\}$$

and let:

$$\mathsf{wc}(Q, N) = \mathsf{sup}_{\mathbb{D} \in \mathcal{D}_Q^{\leq N}} |Q(\mathbb{D})|$$

$\mathsf{wc}(Q, N)$ **is the worst case: the size of the biggest answer set possible with query Q and databases where each table are bounded by N.**

Worst case examples

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- **Cartesian product**: $Q_2 = R_1(x_1) \wedge R_2(x_2)$ has $\mathsf{wc}(Q_2, N) = N^2$.

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- **Square query**: $Q_\square = R(x_1, x_2) \wedge R(x_2, x_3) \wedge R(x_3, x_4) \wedge R(x_4, x_1)$ has $\mathsf{wc}(Q_\square, N) = N^2$.

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- **Triangle query**: $Q_\Delta = R(x, y) \wedge S(x, z) \wedge T(y, z)$, $\mathsf{wc}(Q_\Delta, N) = N^{1.5}$.
- **The n-cycle**: $Q_{C_n}(x_1, ..., x_n) = R_1(x_1, x_2) \wedge R_2(x_2, x_3) \wedge ... \wedge R_n(x_n, x_1)$: $\mathsf{wc}(Q_{C_n}) = N^{\frac{n}{2}}$.

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We know how to compute $\rho(Q)$ such that $\mathsf{wc}(Q, N) = \tilde{O}(N^{\rho(Q)})$ but we do not need it!

This is known as the AGM-bound

Worst case optimal join (WCOJ) algorithms

A join algorithm is **worst case optimal** (wrt $\mathcal{D}_Q^{\leq N}$) if for every Q, $N \in \mathbb{N}$ and $\mathbb{D} \in \mathcal{D}_Q^{\leq N}$, it computes $Q(\mathbb{D})$ in time

$$\tilde{O}(f(|Q|) \cdot \mathsf{wc}(Q, N))$$

- Data complexity model: Q considered constant hence $f(|Q|)$ also.
- In this talk, f will be a reasonable polynomial!

The DBMS approach is not worst case optimal (triangle example from before).

Existing WCOJ Algorithm

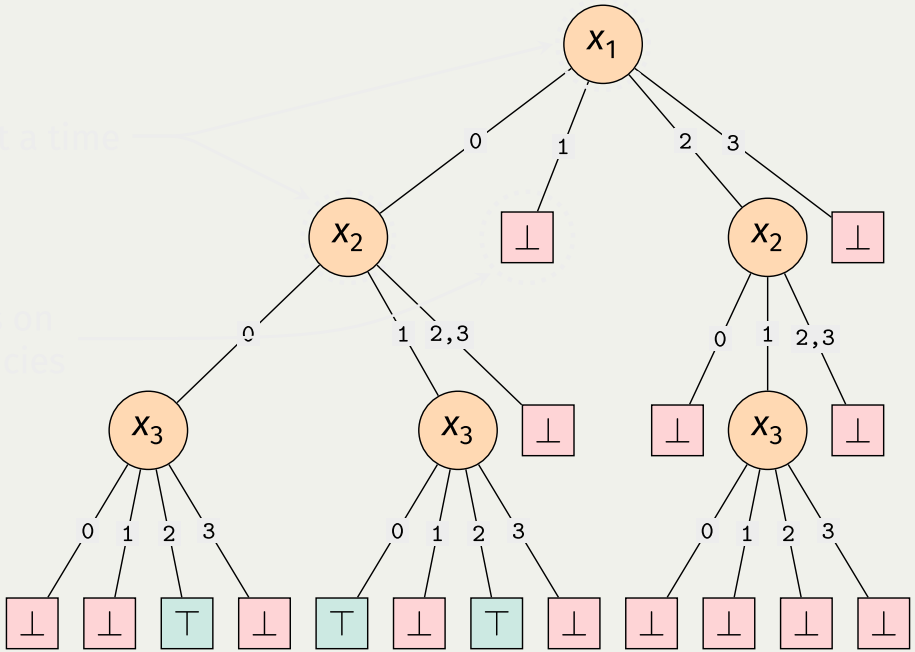
Rich literature:

- **NPRR join** (Ngo, Porat, Ré, Rudra, PODS12): usual join plans but with relations partitionned into high/low degree tuples.
- **Leapfrog Triejoin** (Veldhuizen, ICDT14)
- **Generic Join** (Ngo, PODS18): both branch and bound algorithms as ours but more complex analysis/data structures.
- **PANDA** (PODS17): handle complex database constraints, very complex, long analysis.

We prove the worst case optimality of the branch and bound algorithm in an elementary way.

Analysing the brute

Algorithm reminder



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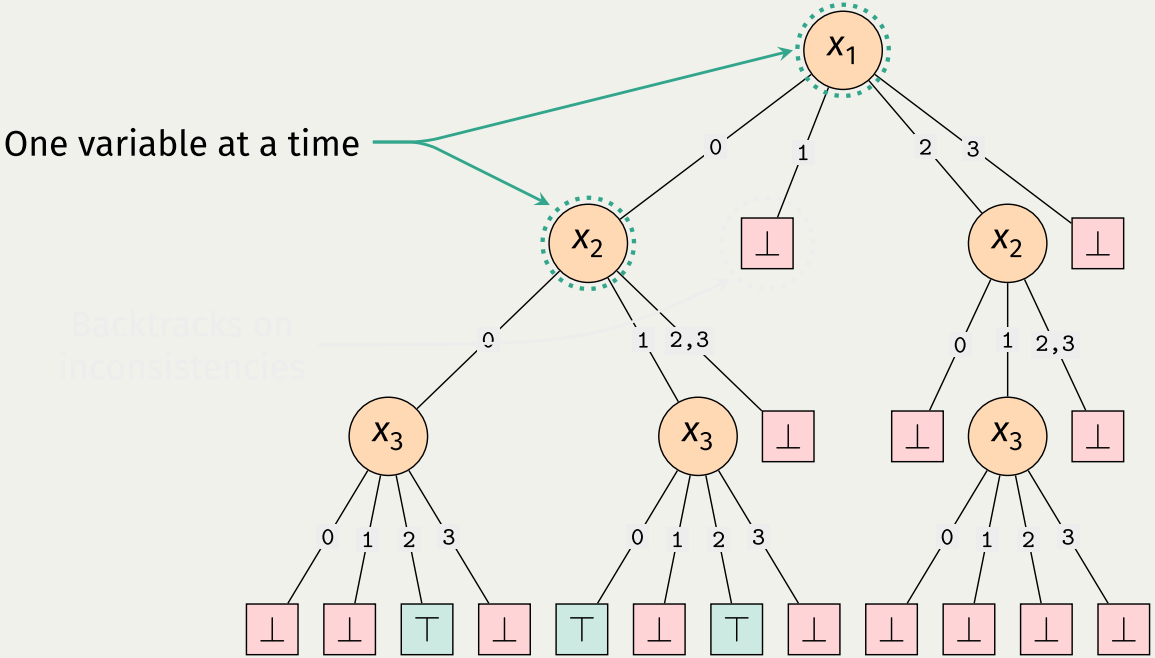

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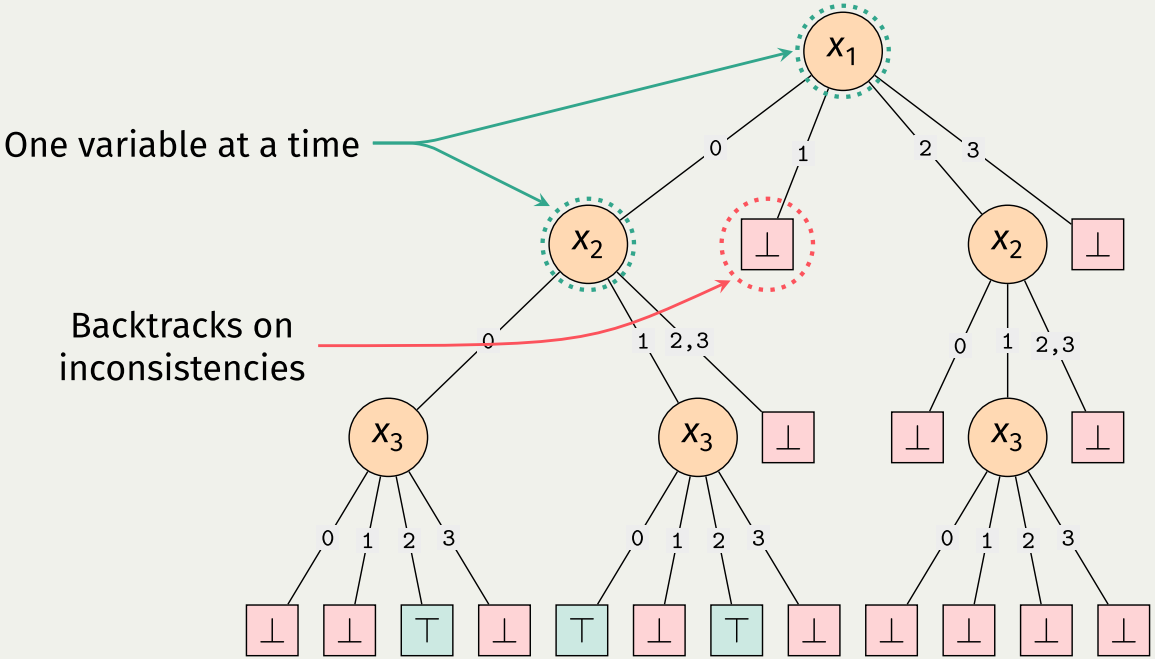
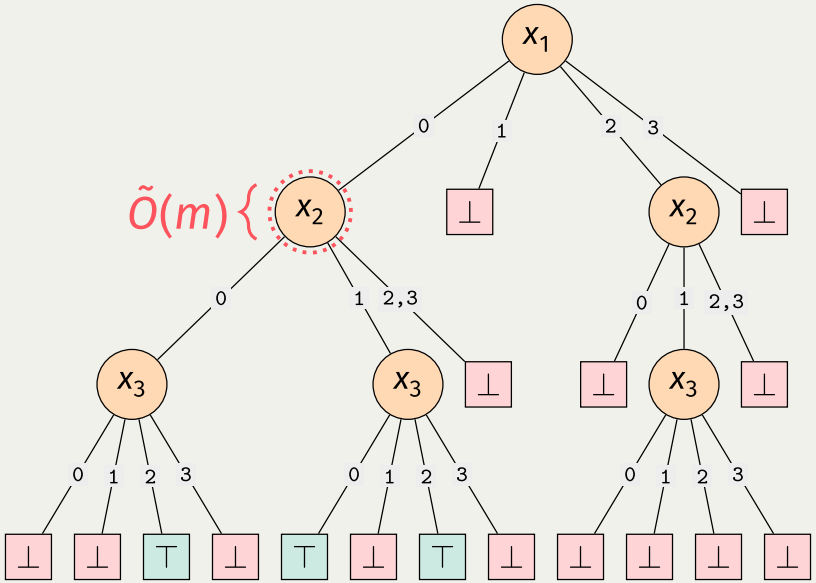


Complexity analysis



One recursive call:

- branch variable x_i on value $d \in \text{dom}$
- filter/project relations with x_i
- Binary search in $O(\log|R|)$ if R ordered ($O(1)$ possible using tries).

R	x_1	x_2	
▶	0	0	
	0	2	
	1	0	
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Total complexity: number of recursive calls times $\tilde{O}(m)$ where m is the number of atoms.

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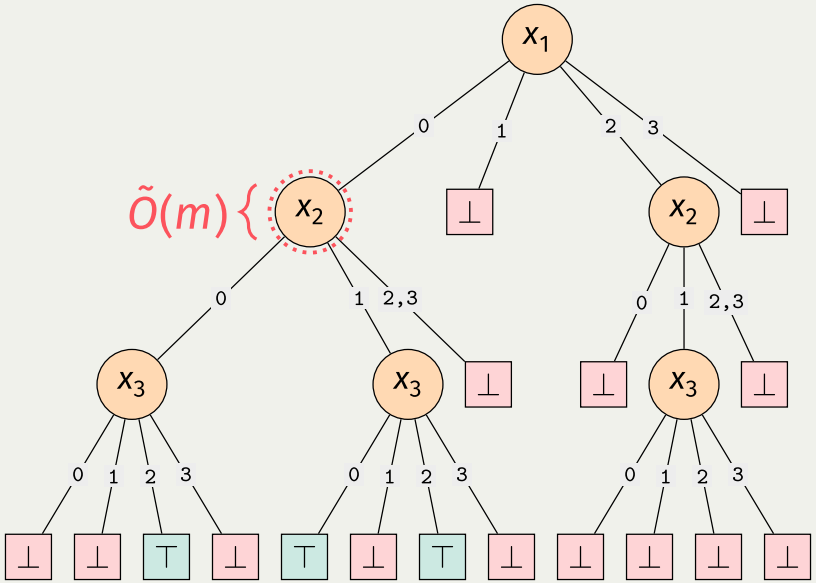

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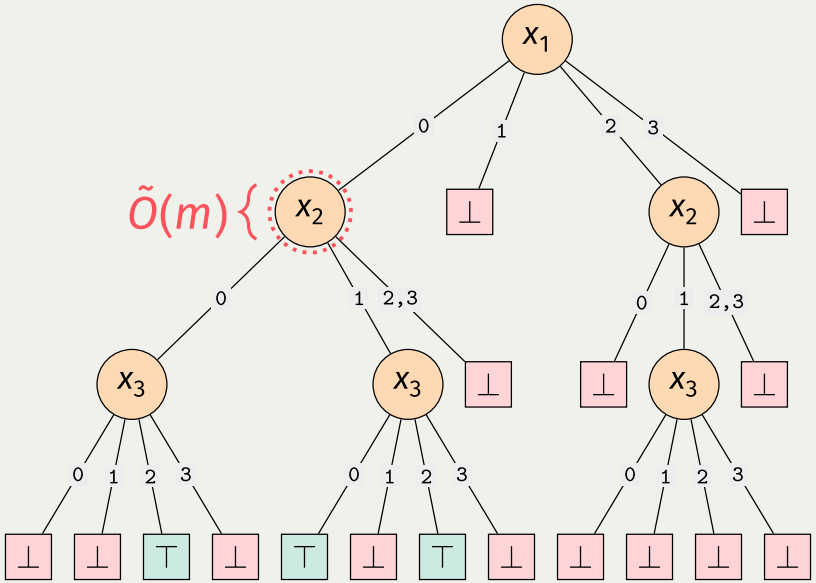

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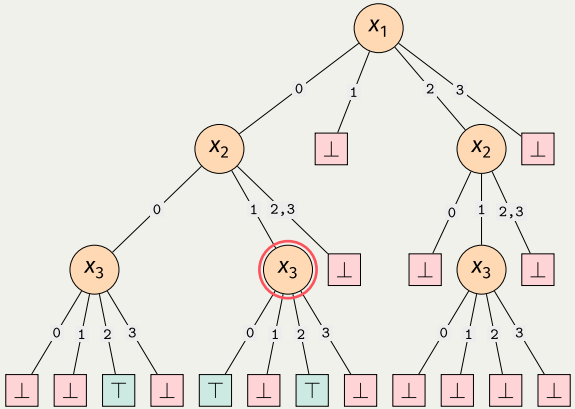
Total complexity: number of recursive calls times $\tilde{O}(m)$ where m is the number of atoms.

Number of calls: example



- Nodes: partial assignment τ
- Here: $\tau := \{x_1 = 0, x_2 = 1\}$
- **Not** $\boxed{\bot}$ node: partial assignment **compatible with every relation**
- τ solution of $Q_2(\mathbb{D}_2)$: project on x_1, x_2.
- At most: $|Q_2(\mathbb{D}_2)|$ such nodes at level 3

$$Q := R(x_1, x_2) \wedge S(x_1, x_3) \wedge T(x_2, x_3)$$

R x_1	x_2
0	0
0	1
2	1

S x_1	x_3
0	0
0	2
2	3

T x_2	x_3
0	2
1	0
1	2

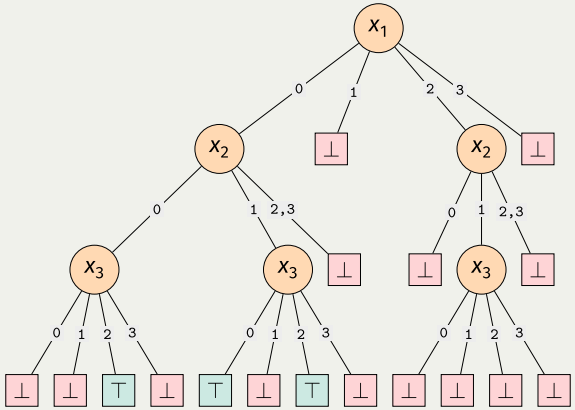
$$Q_2 := R_2(x_1, x_2) \wedge S_2(x_1) \wedge T_2(x_2)$$

R_2 x_1	x_2
0	0
0	1
2	1

S_2 x_1
0
2

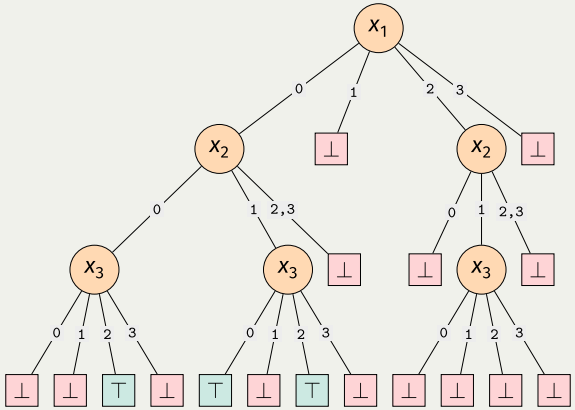
T_2 x_2
0
1

Number of calls in general



- a call = a node = a partial assignment.
- $\tau := x_1 = d_1, ..., x_i = d_i$ current call, not $\boxed{\bot}$:
  - No inconsistency.
  - $R^{\mathbb{D}}[\tau]$ not empty for each $R \in Q$
  - $\tau \in Q_i(\mathbb{D})$ for $Q_i = \bigwedge_{R \in Q} \prod_{x_1 ... x_i} R$
  - $\leq \sum_{i=1}^{n} |Q_i(\mathbb{D})|$ such nodes!

Number of calls in general



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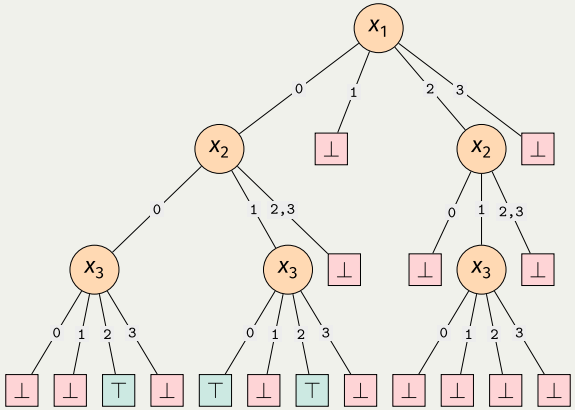
- $\tau := x_1 = d_1, ..., x_{i+1} = d_{i+1}$ current call is $\boxed{\bot}$:
  - $x_1 = d_1, ..., x_i = d_i$ is **not** $\bot$.
  - $\leq |\mathsf{dom}| \cdot \sum_{i=1}^{n} |Q_i(\mathbb{D})|$ $\bot$-nodes!

Number of calls in general



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  - $\leq |\mathsf{dom}| \cdot \sum_{i=1}^{n} |Q_i(\mathbb{D})|$ $\bot$-nodes!

At most $(|\mathsf{dom}|+1) \sum_{i=1}^{n} |Q_i(\mathbb{D})|$ **calls.**

Complexity: $\tilde{O}(m|\mathsf{dom}| \cdot \sum_{i=1}^{n} |Q_i(\mathbb{D})|)$.

Toward worst case optimality

$$|Q_i(\mathbb{D})| = |\bigwedge_{R \in Q} \prod_{x_1 \ldots x_i} R^{\mathbb{D}}|$$
$$= |\bigwedge_{R \in Q} R^{\mathbb{D}'}|$$
$$= |Q(\mathbb{D}')|$$

where $R^{\mathbb{D}'} = \prod_{x_1 \ldots x_i} R^{\mathbb{D}} \times \{0\}^{X_R \setminus x_1, \ldots, x_i}$

$\mathbb{D}$

R x_1	x_2
0	0
0	1
2	1

S x_1	x_3
0	0
0	2
2	3

T x_2	x_3
0	2
1	0
1	2

$\mathbb{D}_2$

R_2 x_1	x_2
0	0
0	1
2	1

S_2 x_1
0
2

T_2 x_2
0
1

$\mathbb{D}'$

R' x_1	x_2
0	0
0	1
2	1

S' x_1	x_3
0	**0**
2	**0**

T' x_2	x_3
0	**0**
1	**0**

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Crucial observation:

- $|R^{\mathbb{D}'}| = |\prod_{x_1 \ldots x_i} R^{\mathbb{D}}| \leq |R^{\mathbb{D}}| \leq N$
- Hence $\mathbb{D}' \in \mathcal{D}_Q^{\leq N}$.
- $|Q_i(\mathbb{D})| = |Q(\mathbb{D}')| \leq \mathsf{wc}(Q, N)$

$\mathbb{D}$

R	x_1	x_2
	0	0
	0	1
	2	1

S	x_1	x_3
	0	0
	0	2
	2	3

T	x_2	x_3
	0	2
	1	0
	1	2

$\mathbb{D}_2$

R_2	x_1	x_2
	0	0
	0	1
	2	1

S_2	x_1
	0
	2

T_2	x_2
	0
	1

$\mathbb{D}'$

R'	x_1	x_2
	0	0
	0	1
	2	1

S'	x_1	x_3
	0	**0**
	2	**0**

T'	x_2	x_3
	0	**0**
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Branch and bound complexity

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We do not even need to know $\mathsf{wc}(Q, N)$ to prove it!

Make the domain binary!

R	x	y
	1	2
	2	1
	3	0

$\rightsquigarrow$

$\tilde{R}^b$	x^2	x^1	x^0	y^2	y^1	y^0
	0	0	1	0	1	0
	0	1	0	0	0	1
	0	1	1	0	0	0

- $Q \rightsquigarrow \tilde{Q}^b$ has bn variables
- $\mathbb{D} \rightsquigarrow \tilde{\mathbb{D}}^b$ for $b = \log|\mathsf{dom}|$. Database has roughly the same bitsize but size 2 domain!

WCOJ finally

- To compute $Q(\mathbb{D})$ run simple branch and bound algorithm on $(\tilde{Q}^b, \tilde{\mathbb{D}}^b)$:
  - runs in time $\tilde{O}(m \cdot (n\log|\mathsf{dom}|) \cdot 2\mathsf{wc}(\tilde{Q}^b, N, 2))$
  - where $\mathsf{wc}(\tilde{Q}^b, N, 2)$ is the worst case for $\tilde{Q}^b$ on relations of size $\leq N$ and domain 2.
  - $\mathsf{wc}(\tilde{Q}^b, N, 2) \leq \mathsf{wc}(Q, N)$ by reconverting back to larger domain.

We hence compute $Q(\mathbb{D})$ in time $\tilde{O}(mn \cdot \mathsf{wc}(Q, N))$!

More on Binarization

- Explicit binarization is not necessary: fix one bit of the variables at a time and filter $\mathbb{D}$.
- Using bigger bases (bytes, words etc.):
  - depth/width tradeoff
  - may be interesting in practice
  - Interesting base: $\log|\mathsf{dom}|$.

Sampling answers uniformly

Problem statement

Given Q and $\mathbb{D}$, sample $\tau \in Q(\mathbb{D})$ with probability $\frac{1}{|Q(\mathbb{D})|}$ or fail if $Q(\mathbb{D}) = \emptyset$.

Naive algorithm:

- materialize $Q(\mathbb{D})$ in a table
- sample $i \leq |Q(\mathbb{D})|$ uniformly
- output $Q(\mathbb{D})[i]$.

Complexity using WCOJ: $\tilde{O}(\mathsf{wc}(Q, N)poly(|Q|))$.

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PODS 23: [Deng, Lu, Tao] and [Kim, Ha, Fletcher, Han]

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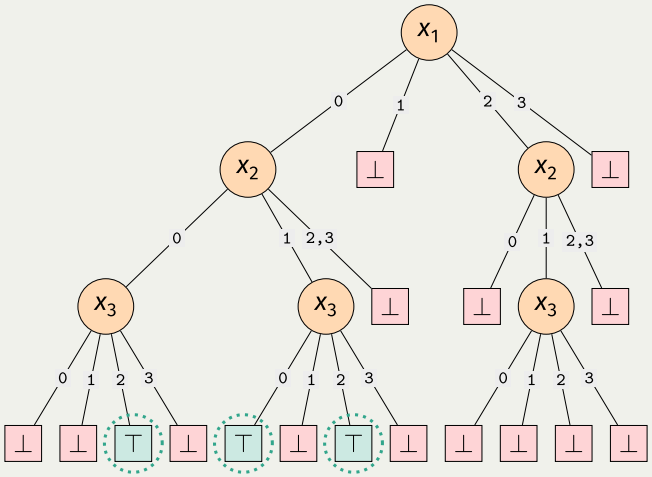
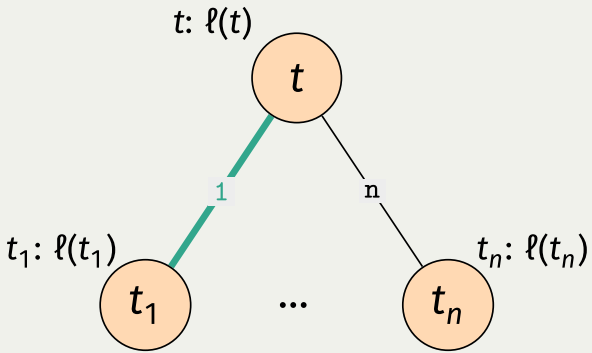
Let's do a modular proof of this fact!

Revisiting the problem



Sampling answers reduces to sampling $\boxed{\top}$-leaves in a tree with ($\boxed{\top}$,$\boxed{\bot}$)-labeled leaves.

Sampling leaves, the easy way

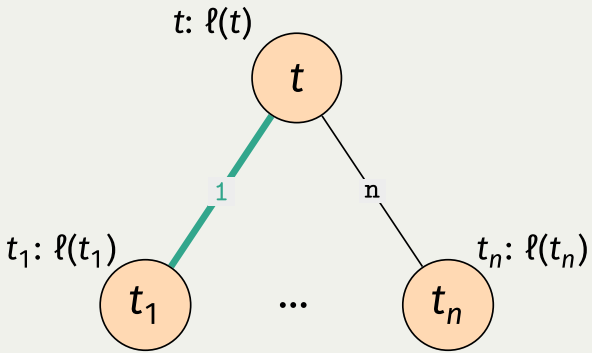


- $\ell(t)$: number of $\boxed{\top}$-leaves below t is known
- Recursively sample uniformly a $\top$-leaf in t_i with probability $\frac{\ell(t_i)}{\ell(t)}$.
- A leaf in $\ell(t_i)$ will hence be sampled with probability

$$\frac{1}{\ell(t_i)} \times \frac{\ell(t_i)}{\ell(t)} = \frac{1}{\ell(t)}$$

Uniform!

Sampling leaves, the easy way



In our case, we do not know $\ell(t)$…

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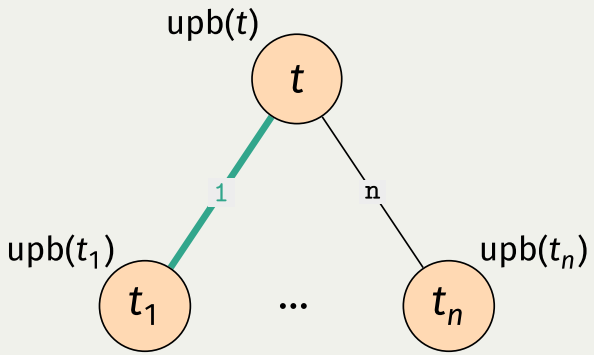
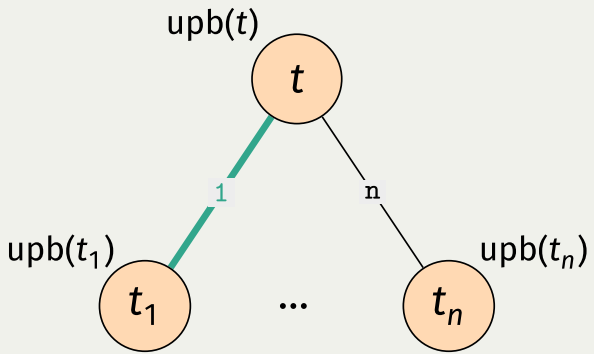
Sampling leaves with a nice oracle



- $upb(t)$: **upperbound** on the number of $\boxed{\top}$-leaves below t is known
- Recursively sample uniformly a $\top$-leaf in t_i with probability $\frac{upb(t_i)}{upb(t)}$.
- **Fail** with probability $1 - \sum_i \frac{upb(t_i)}{upb(t)}$ or upon encountering $\boxed{\bot}$.

Only makes sense if $\sum_i upb(t_i) \leq upb(t)$**.**

Sampling leaves with a nice oracle



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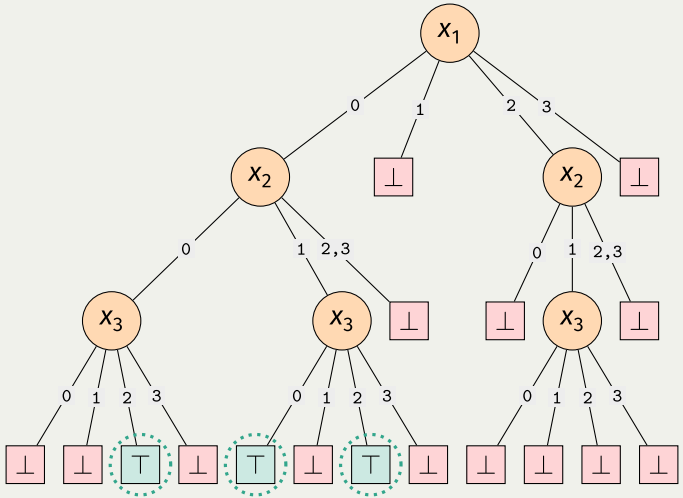
Only makes sense if $\sum_i upb(t_i) \leq upb(t)$**.**

Las Vegas uniform sampling algorithm:

- each leaf is output with probability $\frac{1}{ubp(t)}$,
- fails with proba $1 - \frac{\ell(t)}{upb(t)}$ where $\ell(t)$ is the number of $\boxed{\top}$-leaves under t.

Repeat until output: $O(\frac{upb(r)}{\ell(r)})$ expected calls, where r is the root.

Upper bound oracles for conjunctive queries



- Node t: partial assignment $\tau_t := (x_1 = d_1, ..., x_i = d_i)$
- Number of $\boxed{\top}$ leaves below t: $|Q(\mathbb{D})[\tau_t]|$.
- $upb(t)$???: **look for worst case bounds**!

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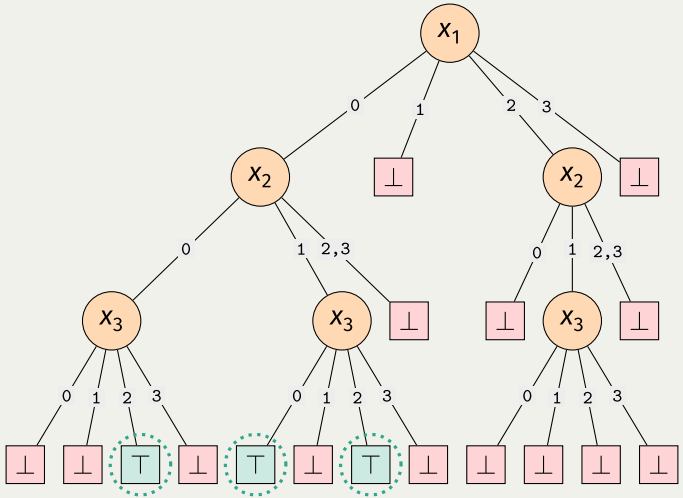

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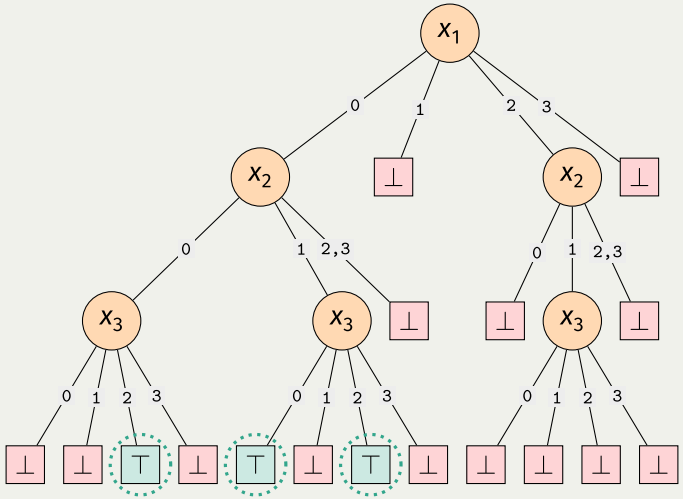
AGM bound: there exists positive rational numbers $(\lambda_R)_{R \in Q}$ such that

$$|Q(\mathbb{D})| \leq \prod_{R \in Q} |R^{\mathbb{D}}|^{\lambda_R} \leq \mathsf{wc}(Q, N)$$

Upper bound oracles for conjunctive queries



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Define $upb(t) = \prod_{R \in Q} |R^{\mathbb{D}}[\tau_t]|^{\lambda_R} \leq \mathsf{wc}(Q, N)$:

- it is an upper bound on $|Q(\mathbb{D})[\tau_t]|$,
- it is supperadditive: $upb(t) \geq \sum_{d \in \mathsf{dom}} upb(t_d)$
- value of upb at the root of the tree: $\mathsf{wc}(Q, N)$!

Wrapping up sampling

Given a super-additive function upperbounding the number of $\boxed{\top}$-leaves in a tree at each node, we have:

Las Vegas uniform sampling algorithm:

- each leaf is output with probability $\frac{1}{ubp(t)}$
- fails with proba $1 - \frac{\ell(t)}{upb(t)}$

Repeat until output: $O(\frac{upb(r)}{\ell(r)})$ expected calls.

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Repeat until output: $O(\frac{upb(r)}{\ell(r)}) = \frac{\mathsf{wc}(Q,N)}{1+|Q(\mathbb{D})|}$ expected calls.

Final complexity: binarize to navigate the tree in $\tilde{O}(nm)$: $\tilde{O}(nm \cdot \frac{\mathsf{wc}(Q,N)}{1+|Q(\mathbb{D})|})$

Matches existing results, proof more modular.

Beyond Cardinality Constraints

Worst case and constraints

So far we have considered worst case wrt this class:

- $\mathcal{D}_Q^{\leq N} = \{\mathbb{D} \mid \forall R \in Q, |R^{\mathbb{D}}| \leq N\}$
- $\mathsf{wc}(Q, N) = \sup_{\mathbb{D} \in \mathcal{D}_Q^{\leq N}} |Q(\mathbb{D})|$

Each relation is subject to a cardinality constraint of size N.

What if we know that our instance has some extra properties (e.g., a *functional dependency*)

- We know $\mathbb{D} \in \mathcal{C} \subseteq \mathcal{D}_Q^{\leq N}$
- We want the join to run in $\tilde{O}(f(|Q|) \cdot \mathsf{wc}(Q, \mathcal{C}))$ where $\mathsf{wc}(Q, \mathcal{C}) := \sup_{\mathbb{D} \in \mathcal{C}} |Q(\mathbb{D})|$.

In this case, we say that our algorithm is worst case optimal wrt $\mathcal{C}$.

Finer constraints can help

$Q = R(x_1, x_2) \wedge S(x_2, x_3)$.

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- Let $\mathcal{C}$ be the class of databases where $|R| \leq N, |S| \leq N$ and R respect functional dependency $x_2 \to x_1$.
- $\mathsf{wc}(Q, \mathcal{C}) \leq N$ because each tuple of $S^{\mathbb{D}}$ can be extended to *at most one solution*.

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Is our simple join worst case optimal for this class?

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- $\mathsf{wc}(Q, \mathcal{C}) \leq N$ because each tuple of $S^{\mathbb{D}}$ can be extended to *at most one solution*.

Is our simple join worst case optimal for this class?

Short answer: yes if x_2 is set before x_1.

Prefix closed classes

Recall the complexity of our algorithm: $\tilde{O}(m|\mathsf{dom}|\sum_{i=1}^{n}|Q_i(\mathbb{D})|))$ where $Q_i = \bigwedge_{R \in Q} \prod_{x_1,\ldots,x_i} R$

A class of database $\mathcal{C}$ for Q is **prefix closed for order** $\pi = (x_1, \ldots, x_n)$ if for each i and $\mathbb{D} \in \mathcal{C}$:

$$|Q_i(\mathbb{D})| \leq \mathsf{wc}(\mathcal{C})$$

$\mathcal{D}_Q^{\leq N}$ **is prefix closed (for any order)!**

Our algorithm is (almost) worst case optimal as long as we use an order for which $\mathcal{C}$ is prefix closed!

Acyclic functional dependencies

$F = (X_1 \to Y_1, ..., X_k \to Y_k)$ is a set of functional dependencies:

- $G(F)$: vertices are the variables and $x \to y$ if $x \in X_i$ and $y \in Y_i$ for some i.
- If $G(F)$ is acyclic, then let $\pi = x_1, ..., x_n$ be a topological sort of $G(F)$. Then

$$\mathcal{C}_F^N = \{\mathbb{D} \mid \mathbb{D} \text{ respects } F\} \cap \mathcal{D}_Q^{\leq N}$$

is **prefix closed for order** π (exactly the same proof as for cardinality constraints).

Hence our algorithm is worst case optimal wrt $\mathcal{C}_F^N$ (as long as we follow π).

We need to show that this functional dependencies transfer in the binarised setting but it is almost immediate.

Degree constraints

A **degree constraint** is a constraint $(X, Y, N_{Y|X})$ where $X \subseteq Y$. A relation R verifies the constraint if

$$\max\{|\prod_{Y} R[\tau]|, \tau \in \mathsf{dom}^X\} \leq N_{Y|X}$$

- Cardinality constraint = degree constraint with $X = \emptyset$.
- Functional dependency = degree constraint with $N_{Y|X} = 1$.

Acyclic degree constraints

$\Delta = \{(X_1, Y_1, N_1)..., (X_k, Y_k, N_k)\}$ set of degree constraints.

- $G(\Delta)$: vertices are the variables and $x \to y$ if $x \in X_i$ and $y \in Y_i$ for some i.
- If $G(\Delta)$ is acyclic, then let $\pi = x_1, ..., x_n$ be a topological sort of $G(\Delta)$. Then

$$\mathcal{C}_\Delta^N = \{\mathbb{D} \mid \mathbb{D} \text{ respects } \Delta\} \cap \mathcal{D}_Q^{\leq N}$$

is **prefix closed for order** π (exactly the same proof as for cardinality constraints).

Hence our algorithm is worst case optimal wrt $\mathcal{C}_\Delta^N$ (as long as we follow π).

We need to show that this functional dependencies transfer in the binarised setting but it is almost immediate.

Bonus: sampling acyclic degree constraints

We can find (λ_R) such that $\prod_{R \in Q} |R^{\mathbb{D}}|^{\lambda_R} \leq \tilde{O}(\mathsf{wc}(Q, \mathcal{C}^N_\Delta))$ for any $\mathbb{D} \in \mathcal{C}^N_\Delta$ (**polymatroid bound**).

Define $upb(t) := \prod_{R \in Q} |R^{\mathbb{D}}[\tau_t]|^{\lambda_R}$:

- upperbound of $Q(\mathbb{D})[\tau_t]$ for any $\mathbb{D} \in \mathcal{C}^N_\Delta$,
- superadditive.

We have sampling with complexity $\tilde{O}(nm \cdot \frac{\mathsf{wc}(Q, \mathcal{C}^N_\Delta)}{1 + |Q(\mathbb{D})|})$

Conclusion

- Simple algorithms and analysis
- Modular:
  - join is worst-case optimal as soon as the class is prefix closed
  - sampling is in $\frac{\mathsf{wc}(Q,\mathcal{C})}{|Q(\mathbb{D})|}$ as long as one can provide a super additive upper bound

Future work:

- Other classes such as:
  - cyclic FD,
  - general system of degree constraints (as PANDA)
- Explore dynamic ordering: can we capture more classes?